Robust Eigenvalue Assignment with Maximum Tolerance to System Uncertainties

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For a linear time-invariant system with a feedback controller, the closed-loop eigenvalues perturb due to system uncertainties. Given an allowable tolerance for the closed-loop eigenvalue perturbation, an algorithm is developed to obtain a state feedback controller that maximizes the uncertainty tolerance of the open-loop system matrix. The design procedure is based on an existing eigenvalue assignment technique using Sylvester's equation. A robustness condition is derived to guarantee satisfaction of a specified closed-loop perturbation tolerance. Finally, an iterative algorithm is presented for easy numerical implementation to compute the robust controller, and a numerical example is given for illustration.

Nomenclature

A = open-loop state matrix
$\tilde{A}$ = matrix with desired closed-loop eigenvalues
B = control input matrix
E = perturbation structure in A
F = state feedback matrix
G = free parameter matrix
I = identity matrix
J = optimization index or cost function
$\bar{u}$ = left singular vector corresponding to $\bar{\sigma}(\cdot)$
$\bar{v}$ = right singular vector corresponding to $\bar{\sigma}(\cdot)$
X = solution of Sylvester's equation
Λ = self-conjugate set of desired eigenvalues
$\lambda(\cdot)$ = set of eigenvalues of matrix $(\cdot)$
$\bar{\sigma}(\cdot)$ = maximum singular value of matrix $(\cdot)$
$\underline{\sigma}(\cdot)$ = minimum singular value of matrix $(\cdot)$
$\|\cdot\|_2$ = spectral norm of matrix $(\cdot)$

I. Introduction

THE eigenvalue assignment problem by state feedback is one of the most fundamental problems in linear control theory. For a given controllable system, the problem is to find a state feedback gain matrix to move the open-loop eigenvalues to the desired closed-loop eigenvalues. Since the pioneering work by Wonham,[1] who used controllable canonical forms, numerous publications have followed. Some of these algorithms are only for the case of a single-input system or to reduce the multi-input problem to a single-input system. For more than a decade, improvements have been made to treat multi-input problems directly and to include additional requirements such as robustness and minimum norm gains by manipulating the nonuniqueness of the eigenvalue assignment solution.

For many practical problems, exact eigenvalue assignment may not be necessary, and any location in some neighborhood of the desired ones is usually acceptable as long as they satisfy the performance requirements. The model of the open-loop system often contains parameter uncertainties due to modeling errors, time- or environment-dependent parameters, etc. For a given state feedback gain that assigns the closed-loop eigenvalues at the exact desired locations, perturbations in the open-loop matrix may cause the closed-loop eigenvalues to move away from the desired locations. It may be useful, and certainly interesting, to determine the maximum allowable parameter perturbations that still guarantee satisfaction of a tolerance requirement given for the closed-loop eigenvalues. This may be obtained by searching over the entire family of state feedback gains that assign the closed-loop eigenvalues at the desired locations. Of course, this is useful only for the case of a multi-input system since the gain is unique for a single-input system.

In this paper, a novel algorithm is developed for obtaining a state feedback controller for a given linear time-invariant system that guarantees the closed-loop eigenvalues to remain within a specified neighborhood while allowing maximum perturbations in the open-loop system matrix. Although perturbations may also occur in the open-loop influence matrix, they are not considered in this paper. In addition, it is assumed that the state feedback matrix is computed accurately and not subjected to perturbation. The approach is to improve iteratively the margin of parameter variations within the desired closed-loop eigenvalue tolerance. The proposed design procedure is based on the eigenvalue assignment algorithm given in Ref. 2. In Sec. II, an effective eigenvalue assignment algorithm[2] is described briefly. In Sec. III, a novel robustness condition is derived to guarantee a specified closed-loop perturbation tolerance. In Sec. IV, an iterative algorithm is outlined for obtaining a robust controller that satisfies the robustness condition and maximizes the tolerance of the open-loop system matrix. In Sec. V, a numerical example is provided for illustration, including a discussion of numerical results. Finally, a few concluding remarks are given.

II. Eigenvalue Assignment via Sylvester's Equation

In this section, the eigenvalue assignment algorithm using Sylvester's equation[2] is described briefly. For a given control-

Presented as Paper 89-3611 at the AIAA Guidance, Navigation, and Control Conference, Boston, MA, Aug. 14-16, 1989; received Dec. 11, 1989; revision received April 24, 1990.

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lable pair (A,B), an observable pair $(G,\tilde{A})$, and a self-conjugate set of n complex numbers Λ, the algorithm assumes the following conditions:

$$\lambda(\tilde{A}) = \Lambda \tag{1}$$

$$\lambda(\tilde{A}) \cap \lambda(A) = \emptyset \tag{2}$$

and

$$\tilde{A} \text{ is cyclic} \tag{3}$$

where $\lambda(\cdot)$ denotes the eigenvalues of $(\cdot)$. A cyclic matrix contains a minimal polynomial[3] identical to its characteristic polynomial.

Under the preceding assumptions on $(A,B,\tilde{A})$, the unique solution X of the following Sylvester's equation is generally nonsingular with respect to the parameter matrix G.

$$AX - X\tilde{A} = -BG \tag{4}$$

Equivalently, the unique solution X of Eq. (4) is nonsingular for every G except those lying in a proper algebraic variety. For every G outside this variety,

$$\lambda(A + BGX^{-1}) = \lambda(\tilde{A}) \tag{5}$$

which gives a state feedback gain

$$F = GX^{-1} \tag{6}$$

for eigenvalue assignment by solving Eq. (4). The algorithm provides a complete parameterization of the family of F in terms of the free parameter matrix G. Several choices of numerically stable algorithms, such as singular-value decomposition,[4] the Bartels-Stewart algorithm,[5] or the Hessenberg-Schur method,[6] are available to solve the preceding equations. This parameterization was used to address a robust eigenstructure problem[7] and a minimum norm state-feedback problem.[8]

III. Robustness Condition

Assume that the given controllable system (A,B) and an appropriate choice of $(G,\tilde{A})$ [$\tilde{A}$ contains the desired closed-loop eigenvalues and $(G,\tilde{A})$ is an observable pair with an almost arbitrary choice of G] satisfy Eq. (4). Let the open-loop matrix A contain the parameter perturbations, which can be expressed as

$$A + \Delta A := A + \epsilon_1 E_1 + \epsilon_2 E_2 + \cdots + \epsilon_r E_r \tag{7}$$

where ϵ_i and E_i represent the magnitude and the perturbation structure of the ith parameter, respectively.

The perturbation ΔA about $A + BF$ can be related to the perturbation $\Delta\tilde{A}$ about $\tilde{A}$ through the same similarity transformation matrix, X, as in Eq. (4), such that

$$\left(A + \sum_{j=1}^{r} \epsilon_j E_j\right) X - X(\tilde{A} + \Delta\tilde{A}) = -BG \tag{8}$$

Combining Eqs. (4) and (8) yields

$$\left(\sum_{j=1}^{r} \epsilon_j E_j\right) X - X\Delta\tilde{A} = 0 \tag{9}$$

or

$$\sum_{j=1}^{r} \epsilon_j X^{-1} E_j X = \Delta\tilde{A} \tag{10}$$

Note that $\Delta\tilde{A}$ represents the perturbation of the closed-loop state matrix in a coordinate system whose basis vectors are the columns of X. Taking the spectral norm of both sides of Eq. (10) leads to

$$\left\| \sum_{j=1}^{r} \epsilon_j X^{-1} E_j X \right\|_2 = \|\Delta\tilde{A}\|_2 \leq \sum_{j=1}^{r} |\epsilon_j| \; \|X^{-1}E_jX\|_2 \tag{11}$$

Defining μ_j and ρ as

$$\mu_j := \|X^{-1}E_jX\|_2 \tag{12a}$$

$$\rho := \|\Delta\tilde{A}\|_2 \tag{12b}$$

and substituting them into the preceding equation gives

$$\begin{aligned} \rho &\leq \sum_{j=1}^{r} |\epsilon_j| \mu_j \\ &= \underbrace{[|\epsilon_1| \; |\epsilon_2| \; \cdots \; |\epsilon_r|]}_{\epsilon} \underbrace{\begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_r \end{bmatrix}}_{\mu} \\ &= \epsilon\mu \end{aligned} \tag{13}$$

Now take the vector norm of ρ

$$\begin{aligned} \rho^2 &\leq \|\epsilon\|_2^2 \; \|\mu\|_2^2 \\ &= \left(\sum_j \epsilon_j^2\right)\left(\sum_j \mu_j^2\right) \end{aligned} \tag{14}$$

which results in

$$\frac{\rho^2}{\sum_j \epsilon_j^2} \leq \sum_j |\mu_j|^2 \tag{15a}$$

$$\frac{\|\Delta\tilde{A}\|_2^2}{\sum_j \|X^{-1}E_jX\|_2^2} \leq \sum_j \epsilon_j^2 \tag{15b}$$

Before a robustness condition is derived, first observe the following lemmas.

Lemma 1. A Jordan matrix $\tilde{A}$ with its ith diagonal block $\tilde{A}_i$ having the structure

$$\begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \quad \text{for all } i \tag{16}$$

is normal.

Lemma 2.[3] A complex square matrix is unitary similar to a complex diagonal matrix if, and only if, it is normal.

Lemma 3.[9] Let $H^{-1}\tilde{A}H = \text{diag}(\lambda_i)$, then for any $\Delta\tilde{A}$, every eigenvalue of $(\tilde{A} + \Delta\tilde{A})$ lies in at least one of the circular disks given by

$$|\lambda_i - \lambda| \leq \|H^{-1}\|_2 \; \|H\|_2 \; \|\Delta\tilde{A}\|_2 \tag{17}$$

where λ is an eigenvalue of $\tilde{A} + \Delta\tilde{A}$.

Lemma 4. If $\tilde{A}$ is normal, then every eigenvalue of $(\tilde{A} + \Delta\tilde{A})$ lies in at least one of the circular disks given by

$$|\lambda_i - \lambda| \leq \|\Delta\tilde{A}\|_2 \tag{18}$$

The proof of this lemma is clear from Lemmas 1-3.

Choosing a normal $\tilde{A}$ is always possible whenever the desired closed-loop eigenvalues are distinct. Based on the preceding discussion, a robustness condition can be derived.

Theorem 1. For the given controllable pair (A,B), let $\tilde{A}$ contain the desired closed-loop eigenvalues, and for an appropriate choice of G, the state feedback F is given by the eigenvalue assignment algorithm shown in Ref. 2. Let ϵ_j for $j = 1, \ldots, r$ denote the parameters of perturbation in A, and the maximum tolerance in the perturbation of the closed-loop eigenvalues be given by

$$\rho = \|\Delta\tilde{A}\|_2 \tag{19}$$

If the total parameter variation $\sum_j \epsilon_j^2$ satisfies

$$\sum_j \epsilon_j^2 \le \frac{\|\Delta\tilde{A}\|_2^2}{\sum_j \|X^{-1}E_jX\|_2^2} \tag{20}$$

then the closed-loop eigenvalues are guaranteed to remain inside the given tolerance centered around the desired values.

Equation (20) defines the robustness condition used in this paper. It is clear that, by a judicious choice of a state feedback F, and equivalently X, one can improve the right-hand-side expression in Eq. (20) to obtain a larger parameter perturbation tolerance while maintaining the same closed-loop eigenvalue tolerance. An iterative procedure is formulated in the sequel to obtain such a state feedback that can tolerate a large-parameter variation.

IV. Iterative Algorithm for Robust Controller

To enlarge the guaranteed parameter perturbation tolerance, consider the following index to be minimized:

$$J := \frac{1}{\|\Delta\tilde{A}\|_2^2}\sum_j \|X^{-1}E_jX\|_2^2 \tag{21}$$

Since $\|\Delta\tilde{A}\|_2$ is a given constant quantity (closed-loop eigenvalue tolerance), the index J is minimized by minimizing

$$J_c := \sum_j \|X^{-1}E_jX\|_2^2 = \sum_j \bar{\sigma}_j^2(X^{-1}E_jX) \tag{22}$$

where $\bar{\sigma}_j(X^{-1}E_jX)$ represents the maximum singular value of $X^{-1}E_jX$. The closed-form expression of its gradient with respect to G, and in turn G, can be easily computed. Indeed, it is known using singular-value decomposition that

$$X^{-1}E_jX = \sum_i u_{ij}v_{ij}^T\sigma_{ij} \tag{23}$$

and thus for the jth parameter,[10]

$$\Delta\bar{\sigma}_j(X^{-1}E_jX) = \bar{u}_j^T\Delta(X^{-1}E_jX)\bar{v}_j \tag{24}$$

where $\bar{v}_j$ and $\bar{u}_j$ denote the right and left singular vectors corresponding to $\bar{\sigma}_j$. Furthermore,

$$\begin{aligned}\Delta(X^{-1}E_jX) &= \Delta(X^{-1})E_jX + X^{-1}E_j\Delta(X)\\ &= -X^{-1}\Delta(X)X^{-1}E_jX + X^{-1}E_j\Delta(X)\end{aligned} \tag{25}$$

Since $\sigma(\cdot)$ is a scalar quantity, the index J_c becomes

$$J_c = \mathrm{tr}\left\{\sum_j \bar{\sigma}_j^2(X^{-1}E_jX)\right\} \tag{26}$$

and thus

$$\begin{aligned}\Delta J_c &= 2\,\mathrm{tr}\left\{\sum_j \bar{\sigma}_j(X^{-1}E_jX)\Delta\bar{\sigma}_j(X^{-1}E_jX)\right\}\\
&= 2\,\mathrm{tr}\left\{\sum_j \bar{\sigma}_j(X^{-1}E_jX)\bar{v}_j\bar{u}_j^T\Delta(X^{-1}E_jX)\right\}\\
&= 2\,\mathrm{tr}\left\{\sum_j \bar{\sigma}_j(X^{-1}E_jX)\bar{v}_j\bar{u}_j^T \times[X^{-1}E_j\Delta(X) - X^{-1}\Delta(X)X^{-1}E_jX]\right\}\\
&= 2\,\mathrm{tr}\left\{\sum_j \bar{\sigma}_j(X^{-1}E_jX)\bar{v}_j\bar{u}_j^TX^{-1}E_j\Delta(X)\right\}\\
&\quad - 2\,\mathrm{tr}\left\{\sum_j \bar{\sigma}_j(X^{-1}E_jX)X^{-1}E_jX\bar{v}_j\bar{u}_j^TX^{-1}\Delta(X)\right\}\\
&= 2\,\mathrm{tr}\left\{\sum_j \bar{\sigma}_j(X^{-1}E_jX)(\bar{v}_j\bar{u}_j^TX^{-1}E_j - X^{-1}E_jX\bar{v}_j\bar{u}_j^TX^{-1})\Delta(X)\right\}\end{aligned} \tag{27}$$

Note that the solution X of the Sylvester's equation [Eq. (4)] is known as

$$X = \sum_k\sum_l \gamma_{kl}A^{k-1}BG\tilde{A}^{l-1} \tag{28}$$

where

$$\Gamma := \begin{bmatrix}\gamma_{11} & \cdots & \cdots & \gamma_{1n}\\ \vdots & & & \vdots\\ \vdots & & & \vdots\\ \gamma_{n1} & \cdots & \cdots & \gamma_{nn}\end{bmatrix}$$

is the unique solution of

$$K_\alpha^T\Gamma - \Gamma K_\beta = [1\ 0\ \cdots\ 0]^T\,[1\ 0\ \cdots\ 0]$$

in which K_α^T and K_β are the companion forms of matrices A and $\tilde{A}$, respectively. The proof of this expression may be found in Ref. 11. Consider a perturbation in Eq. (4) due to G,

$$A[X + \Delta(X)] - [(X + \Delta(X)]\tilde{A} = -B[G + \Delta(G)] \tag{29}$$

and, equivalently,

$$A\Delta(X) - \Delta(X)\tilde{A} = -B\Delta(G) \tag{30}$$

From Eq. (28), the solution $\Delta(X)$ of Eq. (30) becomes

$$\Delta(X) = \sum_k\sum_l \gamma_{kl}A^{k-1}B\Delta(G)\tilde{A}^{l-1} \tag{31}$$

Using Eq. (31) to rewrite Eq. (27) in terms of $\Delta(G)$ yields

$$\begin{aligned}\Delta J_c &= 2\,\mathrm{tr}\left\{\sum_j\sum_k\sum_l \gamma_{kl}\bar{\sigma}_j(X^{-1}E_jX) \times(\bar{v}_j\bar{u}_j^TX^{-1}E_j - X^{-1}E_jX\bar{v}_j\bar{u}_j^TX^{-1})A^{k-1}B\Delta(G)\tilde{A}^{l-1}\right\}\\
&= 2\,\mathrm{tr}\Bigg\{\sum_k\sum_l \gamma_{kl}\tilde{A}^{l-1}\underbrace{\sum_j \bar{\sigma}_j(X^{-1}E_jX)}_{Z}\\
&\qquad \times\underbrace{(\bar{v}_j\bar{u}_j^TX^{-1}E_j - X^{-1}E_jX\bar{v}_j\bar{u}_j^TX^{-1})}_{Z\text{(continued)}}A^{k-1}B\Delta(G)\Bigg\}\\
&= 2\,\mathrm{tr}\Bigg\{\underbrace{\sum_k\sum_l \gamma_{kl}\tilde{A}^{l-1}ZA^{k-1}}_{Y}B\Delta(G)\Bigg\}\end{aligned} \tag{32}$$

From Eqs. (30) and (31), it is clear that Y is the unique solution of the equation

$$\tilde{A}Y - YA = -Z \tag{33}$$

Therefore,

$$\Delta J_c = 2\,\mathrm{tr}\{YB\Delta(G)\} \tag{34}$$

where Y satisfies

$$\tilde{A}Y - YA = -\sum_j \bar{\sigma}_j (X^{-1}E_jX) \times \{\bar{v}_j\bar{u}_j^T X^{-1}E_j - X^{-1}E_jX\bar{v}_j\bar{u}_j^T X^{-1}\} \tag{35}$$

Thus, the gradient of J_c with respect to X becomes

$$\frac{\partial J_c}{\partial G} = 2B^TY^T \tag{36}$$

The preceding problem can be viewed as an unconstrained minimization problem

$$\underset{G}{\text{minimize}}\; J_c[X(G)] \tag{37}$$

where X and G satisfy Eq. (4). Although various alternative algorithms exist (see, for example, Ref. 12) for solving the preceding nonlinear programming problem, a useful algorithm is presented in the following, which is based on the discrete Newton method.[13] The algorithm includes several steps.

0) Choose initial G^0 and $\tilde{A}^0$ and compute X^0 from Eq. (4).
1) Compute $\partial J_c/\partial g$ from Eqs. (35) and (36).
2) Compute $\partial^2 J_c/\partial g^2$ from a finite-difference approximation.
3) Solve for update

$$\left[\frac{\partial^2 J_c}{\partial g^2}\right]\Delta g = -\left(\frac{\partial J_c}{\partial g}\right) \tag{38}$$

4) Update $g = g + \Delta g$.
5) Convergence check:
   - IF $\|\partial J_c/\partial g\| \leq \delta$ (prescribed) AND $\partial^2 J_c/\partial g^2 > 0$
     THEN END
     ELSE GOTO step 1
   - ENDIF

In the preceding algorithm,

$$g := \text{vec}(G) \tag{39}$$

For a given tolerance $\|\Delta\tilde{A}\|_2$ and a minimum value of J_c, one can easily compute the corresponding maximum perturbation tolerance of parameters. Although the discrete Newton method is neither the most computationally efficient nor the most reliable technique, the method is relatively easy to implement and it fits our purpose of demonstrating the validity and significance of the robustness measure introduced herein.

V. Illustrative Examples

Consider a simple linear dynamic system of the three lumped mass-spring-dashpot connected in series and fixed at one end (see Fig. 1). This system was used in Ref. 14 as an illustrative example for robust eigensystem assignment for flexible structures.

The equations of motion for the controllable force u_1 acting on mass m_1 and force u_2 acting on mass m_3 can be written as

$$M\ddot{q} + C\dot{q} + Kq = Du, \qquad q^T = [q_1\; q_2\; q_2]$$

or

$$\dot{x} = Ax + Bu; \qquad x^T = [q^T\; \dot{q}^T]$$

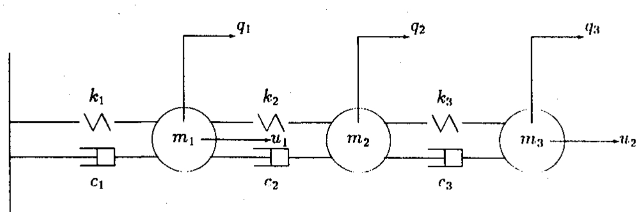

Fig. 1 Three mass-spring-dashpot system.

Table 1 Closed-loop system parameters

Mode	ω	ζ	Closed-loop eigenvalues
1	1.0	0.50	$-0.5 \pm j0.866$
2	2.0	0.50	$-1.0 \pm j1.732$
3	3.0	0.50	$-1.5 \pm j2.598$

where

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ M^{-1}D \end{bmatrix}$$

The mass, damping, stiffness, and force distribution matrices are, respectively, chosen as

$$M = \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix}$$

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Let the nominal values of the system parameters be

$$\begin{array}{lll} m_1 = 1, & m_2 = 1, & m_3 = 1 \\ c_1 = 0, & c_2 = 0, & c_3 = 0 \\ k_1 = 5, & k_2 = 5, & k_3 = 20 \end{array}$$

The desired closed-loop system eigenvalues are presented in Table 1.

Our task is then to design a robust feedback gain, F, when $u = Fx$, and to determine the allowable parameter perturbation $|\epsilon|$ such that each closed-loop eigenvalue must remain inside disks of radius 0.5 centered on the desired eigenvalue locations. Consider the following two cases.

Case 1: If c_3 is the only parameter subject to perturbation, the open-loop matrix becomes

$$A + c_3E_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -10 & 5 & 0 & 0 & 0 & 0 \\ 5 & -25 & 20 & 0 & 0 & 0 \\ 0 & 20 & -20 & 0 & 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

The minimum value of J_c calculated after 19 iterations is

$$J_c = \sum_j \|X^{-1}E_jX\|_2^2 = 4.9231269$$

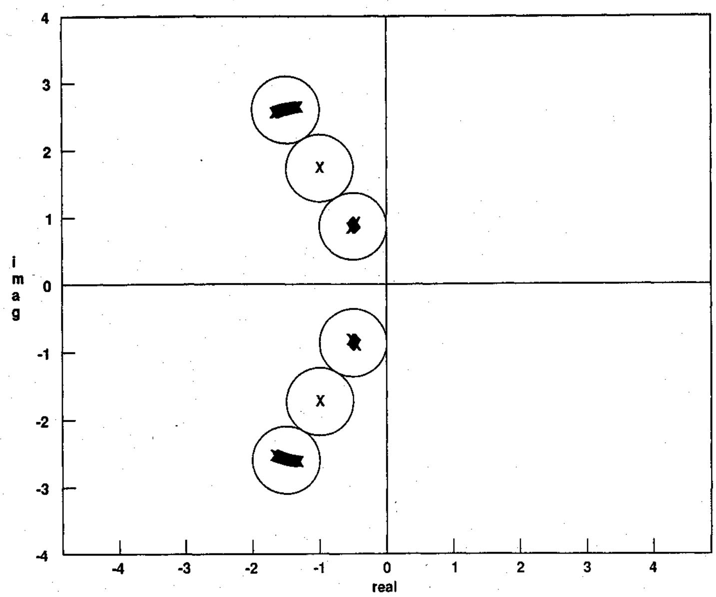

Fig. 2 Trace of closed-loop eigenvalues vs parameter perturbation in c_1.

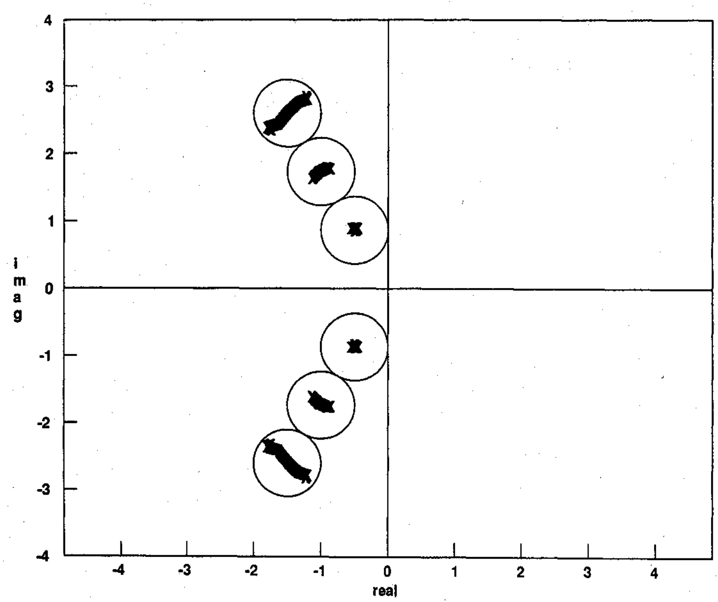

Fig. 3 Trace of closed-loop eigenvalues vs parameter perturbation in c_1, c_2, and c_3.

and the corresponding state feedback matrix is

$$F = \begin{bmatrix} 3.8323 & 13.3411 \\ 10.878 & -66.8645 \\ -11.780 & 55.7871 \\ -4.2903 & 2.75540 \\ 9.5107 & 2.29038 \\ -8.0873 & -1.70964 \end{bmatrix}^T$$

Therefore, if the amount of perturbation in c_3 satisfies

$$|c_3|^2 \leq \frac{\|\Delta\tilde{A}\|_2^2}{\sum_j \|X^{-1}E_jX\|_2^2} = \frac{0.5^2}{4.9231269}$$

the eigenvalues of the closed-loop system are guaranteed to remain within the given tolerance. Figure 2 shows the trace of each closed-loop eigenvalue as c_3 varies within the obtained perturbation tolerance. All eigenvalues stay inside the given tolerance.

Case 2: If c_1, c_2, and c_3 are parameters subject to perturbation, the open-loop matrix becomes

$$A + c_1E_1 + c_2E_2 + c_3E_3$$

The minimum value of J_c computed after 26 iterations is

$$J_c = \sum_j \|X^{-1}E_jX\|_2^2 = 7.81342$$

and the corresponding state feedback matrix is

$$F = \begin{bmatrix} 9.3383 & 3.2009 \\ -17.839 & -32.091 \\ 13.062 & 28.550 \\ -2.1551 & -0.38881 \\ 2.8182 & 3.7189 \\ -1.0225 & -3.8448 \end{bmatrix}^T$$

Therefore, if the amount of perturbation satisfies

$$\sum_{j=1}^{3} |c_j|^2 \leq \frac{\|\Delta \tilde{A}\|_2^2}{\sum_j \|X^{-1}E_j X\|_2^2} = \frac{0.5^2}{7.81342}$$

the eigenvalues of the closed-loop system are guaranteed to remain within the given tolerance. Figure 3 shows the trace of each closed-loop eigenvalue, whereas the arbitrary combined perturbation of c_1, c_2, and c_3 satisfies the given perturbation tolerance.

VI. Conclusion

A new algorithm is introduced to determine a robust state feedback so that the closed-loop eigenvalues remain within an arbitrary neighborhood around their desired locations when bounded parameter perturbations occur in the open-loop matrix. The results are somewhat conservative, and the degree of conservatism depends on the particular nominal plant and its perturbation structure. A possible extension of this problem is to specify an arbitrary tolerance limit for each closed-loop eigenvalue. This problem may be more practical because, in many cases, dominant eigenvalues should have smaller tolerance limits than those for less significant eigenvalues. The formulation thus derived may allow larger tolerances of parameter perturbations than that presented in this paper.

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